Collocation Method

Want to solve the two-point BVP

$$\begin{cases} u'' = f(t,u,u'), & a<t<b \\ u(a)=\alpha, \quad u(b)=\beta \end{cases}$$

Idea: (1) "Pretend" the solution is of the form

$$u(t) \approx \sum_{i=1}^{n} \underset{\uparrow\ \text{coefficients}}{c_i}\ \underset{\uparrow\ \text{"basis functions" defined on } [a,b]}{\phi_i(t)}$$

Possible choices of ϕ_i's:

polynomials, B-splines, trig-functions

(2) Then $u''(t) \approx \sum_{i=1}^{n} c_i \phi_i''(t)$ but also

$$u''(t) = f(t,u,u') \approx f\left(t, \sum_{i=1}^{n} c_i \phi_i(t), \sum_{i=1}^{n} c_i \phi_i'(t)\right)$$

So that

$$\sum_{i=1}^{n} \underset{\text{unknown}}{c_i}\ \underset{\uparrow\ \text{known}}{\phi_i''(t)} \approx \underset{\uparrow\ \text{known}}{f}\left(t, \sum_{i=1}^{n} c_i \underset{\uparrow\ \text{known}}{\phi_i(t)}, \sum_{i=1}^{n} c_i \underset{\uparrow\ \text{known}}{\phi_i'(t)}\right)$$

So we have n-unknowns & we'd like to create n-equation

(3) $\Rightarrow$ define a set of n collocation points

$$a = t_1 < t_2 < \cdots < t_n = b$$

and solve the system:

$$\begin{cases} \sum_{i=1}^{n} c_i \phi_i(t_1) = \alpha \\ \sum_{i=1}^{n} c_i \phi_i''(t_j) \approx f\left(t_j, \sum_{i=1}^{n} \underline{c_i} \phi_i(t_j), \sum_{i=1}^{n} \underline{c_i} \phi_i'(t_j)\right) \quad j = 2, \ldots, n-1 \\ \sum c_i \phi_i(t_n) = \beta \end{cases}$$

$\Rightarrow$ we now have the c_i's

$\Rightarrow$ we now have

$$u(t) \approx \sum_{i=1}^{n} c_i \phi_i(t)$$

Example:

$$\begin{cases} u'' = 6t & 0 < t < 1 \\ u(0) = 0, \quad u(1) = 1 \end{cases}$$

Let's use $t_1 = 0$, $t_2 = 1/2$, $t_3 = 1$

and let's use $\phi_1(t) = 1$, $\phi_2(t) = t$, $\phi_3(t) = t^2$

so that our approx. sol'n

$$w(t) = c_1 + c_2 t + c_3 t^2$$

$$\Rightarrow w'(t) = c_2 + 2c_3 t \ , \quad w''(t) = 2c_3$$

We'd like $\begin{cases} w''(t_i) = 6t_i \ , & i=2 \\ w(0) = 0 \ , \ w(1) = 1 \end{cases}$

$$\Leftrightarrow \begin{cases} 2c_3 = 6(1/2) \\ c_1 + c_2(0) = 0, \ c_1 + c_2^{\times 1} + c_3^{\times 1^2} = 1 \end{cases}$$

$$\Rightarrow c_1 = 0, \ c_2 = -0.5, \ c_3 = 1.5$$

and $w(t) = 0 + -0.5t + 1.5t^2$

(note that the true sol'n is : $u'' = 6t$

$\Rightarrow u' = 3t^2 + a$

$\Rightarrow u = 8t^3 + at + b$

$u(0) = 0 \Rightarrow b = 0$

$u(1) = 1 \Rightarrow 1 + a = 1$

$\Rightarrow a = 0$

$\Rightarrow u(t) = t^3$)

Example:

$$\begin{cases} u'' + p(t)u' + q(t)u = z(t) \\ u(a) = \alpha \quad , \quad u(b) = \beta \end{cases}$$

Use B-splines $B_i^{k \leftarrow \text{order}}$ $k \geq 3$
($\Rightarrow$ 2 cont's derivatives) (170B)

Choose the knots such that
$t_i = t_{i-1} + h \ , \ t_1 = a$

and use the knots as collocation points.

So, we want to approx. the sol'n $u(t)$ by

$$w(t) = \sum_{j=1}^{n} c_j \phi_j(t)$$

where $\uparrow$ B-splines (chosen well)

$$\boxed{w(t_i) = \sum_{j=1}^{n} c_j \phi_j(t_i)}$$

satisfies

$$\boxed{\begin{cases} w''(t_i) + p(t)\,w'(t_i) + q(t)\,w(t_i) = Z(t_i) \\ \text{for } \quad i=1,\cdots,n-2 \\ \&\ w(a)=\alpha, \quad w(b)=\beta \end{cases}}$$

Using prop's of B-splines this leads to a system of linear equations for the coeff's $\vec{C}=(c_1,\cdots,c_n)$

$$Ac=b \quad \text{where}$$

A is a banded matrix.

$\Rightarrow$ fast inversion is possible.

(170 A)

Collocation for IVP's

want to solve

$$\begin{cases} y'(t) = f(t,y) \\ y(t_0) = \alpha \end{cases}$$

over, say $[t_0, t_0+h]$

Idea: approximate the sol'n $y(t)$ by a polynomial $p(t)$ of degree n

($\Rightarrow$ $n+1$ parameters needed)

Want

$$\begin{cases} p'(t_k) = f(t_k, p(t_k)), \\ p(t_0) = \alpha \end{cases}$$

$\hookrightarrow$ $n+1$ equations

Example ($n=2 \Rightarrow$ trap. rule!)

$$\begin{cases} p(t_0) = y_0 \\ p'(t_0) = f(t_0, p(t_0)) \\ p'(t_0+h) = f(t_0+h, p(t_0+h)) \end{cases}$$

write $p(t) = c_3(t-t_0)^2 + c_2(t-t_0) + c_1$

and solve for c_1, c_2, c_3

$$\Rightarrow \begin{cases} c_1 = y_0 \\ c_2 = f(t_0, p(t_0)) \\ \underbrace{2c_3(h) + c_2}_{p'(t_0+h)} = f(t_0+h, p(t_0+h)) \end{cases}$$

$$\Rightarrow c_3 = \frac{f(t_0+h, p(t_0+h)) - f(t_0, p(t_0))}{2h}$$

$\Rightarrow$ we now have

$$p(t_0+h) = y_0 + f(t_0, p(t_0))h$$
$$+ \frac{f(t_0+h, p(t_0+h)) - f(t_0, p(t_0))}{2h}(h)^2$$

$$\Rightarrow \underbrace{p(t_0+h)}_{y_1} = y_0 + \frac{h}{2}\left[f(t_0+h, p(t_0+h)) + f(t_0, p(t_0))\right]$$

$$= y_0 + \frac{h}{2}\left[f(t_0+h, y_1) + f(t_0, y_0)\right]$$

Implicit eq'n in y_1

solve and repeat!

(to get $y_2, \ldots$)

∎

Stiff Equations

Solving IVP's using numerical methods leads to errors that involve a higher derivative of the solution.

Problems can happen if the magnitude of the derivative increases (but the solution does not) IVP's with this type of issue are called stiff.

Example: Consider the IVP

$$\begin{cases} x_1' = 9x_1 + 24x_2 + 5\cos t - \frac{1}{3}\sin t \\ x_2' = -24x_1 - 51x_2 - 9\cos t + \frac{1}{3}\sin t \\ x_1(0) = \frac{4}{3}, \quad x_2(0) = \frac{2}{3} \end{cases}$$

This has the unique sol'n

$$x_1(t) = \underbrace{2e^{-3t}}_{\text{“slow decay”}} - \underbrace{e^{-39t}}_{\text{“Fast decay”}} + \frac{1}{3}\underbrace{\cos t}_{\text{“oscillatory”}}$$

$$x_2(t) = \overbrace{-e^{-3t}} + \overbrace{2e^{-39t}} - \frac{1}{3}\overbrace{\cos t}$$

Here, the e^{-39t} term causes the equation to be stiff.

(derivatives of $e^{-ct} \to ce^{-ct} \to \cdots \to \underbrace{c^n e^{-ct}}_{!!}$)

How does this affect numerical methods?

In the example above, RK4 with $h=0.05$ works just fine but RK4 with $h=0.1$ blows up $\begin{pmatrix} w_1(t) \to -\infty \\ w_2(t) \to \infty \end{pmatrix}$

Q: How can we predict/understand "stiffness" when seeking numerical solutions?

A: 1) Fix the numerical method

2) Examine the error it produces when applied to the test equation

$$\boxed{\begin{cases} x' = \lambda x \\ x(0) = 1 \end{cases}} \longleftarrow (*)$$

which has the sol'n $x(t) = e^{\lambda t}$

(interested in $\lambda < 0$)

Example: Euler's Method

$$\begin{cases} w_0 = 1 \\ w_{n+1} = w_n + h \underbrace{f(t_n, w_n)}_{\lambda w_n \text{ in } (*)} \end{cases}$$

$$\Rightarrow \quad w_{n+1} = w_n + h\lambda w_{n_\xi}$$

$$= (1+h\lambda) w_n$$

$$\boxed{w_{n+1} = (1+h\lambda)^{n+1} w_0} \longleftrightarrow \text{Euler's method approx.}$$

$$\boxed{\text{True solution at } (n+1)h : \; e^{\lambda(n+1)h}}$$

Reindexing: the absolute error is

$$|x(t_n) - w_n| = |e^{\lambda h n} - \underbrace{(1+h\lambda)^n}_{\text{only decays to zero if } |1+h\lambda|<1 \;!!}|$$

↓ when $\lambda < 0$ this decays to zero

$$\Rightarrow \quad \text{we need} \quad -1 < 1+h\lambda < 1$$

$$\Leftrightarrow -2 < \underset{\text{+ve}}{h}\underset{\text{-ve}}{\lambda} < 0$$

So we need $\boxed{h < -\frac{2}{\lambda}}$ for Euler

In other words bigger $|\lambda|$ requires smaller step size even though the true solution $\longrightarrow 0$ very fast!

<u>Example</u>: Taylor method of order k

Here $w_n = \left(1+h\lambda+\frac{1}{2}h^2\lambda^2+\cdots+\frac{1}{n!}h^k\lambda^k\right)w_{n-1}$

So we'd want

$$\left|\left(1+h\lambda+\frac{1}{2}h^2\lambda^2+\cdots+\frac{1}{n!}h^k\lambda^k\right)\right|<1$$

$\circ\circ\circ$

<u>Example</u>: <u>Implicit Euler method</u>

$$w_{n+1} = w_n + hf(t_{n+1}, w_{n+1})$$

$$\Rightarrow \begin{cases} w_0 = 1 \\ w_{n+1} = w_n + h\lambda w_{n+1} \end{cases} \quad \text{— using } (*)$$

$$\Rightarrow \quad w_{n+1} = (1-h\lambda)^{-1} w_n$$

$$\Rightarrow \quad \boxed{w_n = (1-h\lambda)^{-n} w_0}$$

so now, we want

$$\Big|\underbrace{e^{+\lambda h n}}_{\substack{\to 0 \\ \text{as } n\to\infty}} - \underbrace{(1-h\lambda)^{-n}}\Big| \text{ small}$$

want $|1-h\lambda|^{-1} < 1$

Always true! when $h>0$ $\left(\begin{array}{l} |1-\lambda h|^{-1} < 1 \iff \\ |1-\underset{-ve}{\lambda}\underset{+ve}{h}| > 1 \end{array} \right)$

Systems & Stiffness

Same Idea: Fix a method, examine on a test case

Example:

$\alpha<\beta<0$

$$\begin{cases} x' = \alpha x + \beta y \quad , & x(0) = 2 \\ y' = \beta x + \alpha y \quad , & y(0) = 2 \end{cases}$$

$$\left(\text{True sol'n:} \quad \begin{aligned} x(t) &= e^{(\alpha+\beta)t} + e^{(\alpha-\beta)t} \\ y(t) &= e^{(\alpha+\beta)t} - e^{(\alpha-\beta)t} \end{aligned} \right)$$

<u>Euler's method</u> here would yield

$$\begin{cases} w_{n+1} = w_n + h(\alpha w_n + \beta v_n) \quad , & w_0 = 2 \\ v_{n+1} = v_n + h(\beta w_n + \alpha v_n) \quad , & y_0 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} w_n = (1+\alpha h + \beta h)^n + (1+\alpha h - \beta h)^n \\ v_n = (1+\alpha h + \beta h)^n - (1+\alpha h - \beta h)^n \end{cases}$$

<u>Want w_n & v_n to decay</u>!

So we want $|1+\alpha h + \beta h| < 1$

$|1+\alpha h - \beta h| < 1$

$$\Leftrightarrow \text{ want } \boxed{h < \frac{-2}{\alpha+\beta}}$$

General linear multi-step methods

Recall:
$$\boxed{\sum_{i=0}^{k} a_{k-i} x_{n-i} = h \sum_{i=0}^{k} b_{k-i} f_{n-i}}$$

↳ general multi-step method

Apply this to the test problem

$$\boxed{\begin{cases} x' = \lambda x \\ x(0) = 1 \end{cases}} \longrightarrow (*) \qquad (\lambda < 0)$$

To get
$$\boxed{\sum_{i=0}^{k} a_{k-i} x_{n-i} = \lambda h \sum_{i=0}^{k} b_{k-i} x_{n-i}}$$

$$\Downarrow$$

$$\boxed{(a_k - h\lambda b_k) x_n + \cdots + (a_0 - h\lambda b_0) x_{n-k} = 0}$$

↳ sol'n is a combo of terms: $x_n = r^n$
where r is a root of $\phi(z)$

$$\boxed{\begin{aligned}\phi(z) &= (a_k - h\lambda b_k) z^k + (a_{k-1} - h\lambda b_{k-1}) z^{k-1} \\ &+ \cdots + (a_0 - h\lambda b_0)\end{aligned}}$$

Characteristic polynomial

$$\left(\phi(z) = p(z) - h\lambda\, q(z) \right)$$

from stability section

Fact: In order to obtain a decaying numerical sol'n we need the complex roots of $\phi(z)$ to lie in the disk given by

$$|z| < 1$$

A-Stability

We're been considering $\begin{cases} x' = \lambda x \\ x(0) = 1 \end{cases}$

with $\lambda < 0$

Let us now consider complex

λ : $\boxed{\lambda = \mu + i\nu}$

now, the sol'n to (*) is

$$\boxed{x(t) = e^{\lambda t} = e^{\mu t}(\cos \nu t + i \sin \nu t)}$$

we're interested in $\boxed{\mu < 0}$

So, for a multistep method to do well, we want the roots of $\phi(z)$ to be in the unit disk whenever $h > 0$, $\mu = \mathrm{Re}(\lambda) < 0$

This property is A-stability

Example : * Implicit Euler is A-stable (check)

* Implicit Trapezoid method

$$w_n = w_{n-1} + \frac{1}{2}h\Big[\underbrace{f_n}_{\lambda w_n} + \underbrace{f_{n-1}}_{\lambda w_{n-1}}\Big]$$ is

also A-stable, bec.

$$\phi(z) = z - 1 - \frac{1}{2}h\lambda(z+1)$$

$$= \left(1-\frac{1}{2}h\lambda\right)z - \left(1+\frac{1}{2}h\lambda\right)$$

has root $$z^* = \frac{1+h\lambda/2}{\underbrace{1-h\lambda/2}}$$

when $h>0$ & $Re(\lambda)<0$

$$|z^*| = \left|\frac{2+\lambda h(\mu+i\nu)}{2-h\lambda(\mu+i\nu)}\right| < 1$$

Theorem : Among linear multi-step methods, only implicit method of order ≤ 2 can be A-stable

Region of Absolute stability (for multi-step methods)

Idea: want roots of

$\phi(z) = p(z) - h\lambda q(z)$ to be in unit disk, so the multi-step method can work on the test prob.

$x' = \lambda x$

So, we are interested in:

Region of absolute stability:

$\{ w \in \mathbb{C} : \text{roots of } p(z) - w q(z)$ lie in the interior of the unit disk $\}$

(A-stable methods work for all $h > 0$, other methods work when h is small enough)

Example: $x_n = x_{n-1} + hf_{n-1}$ $\quad f_{n-1} = \lambda x_{n-1}$

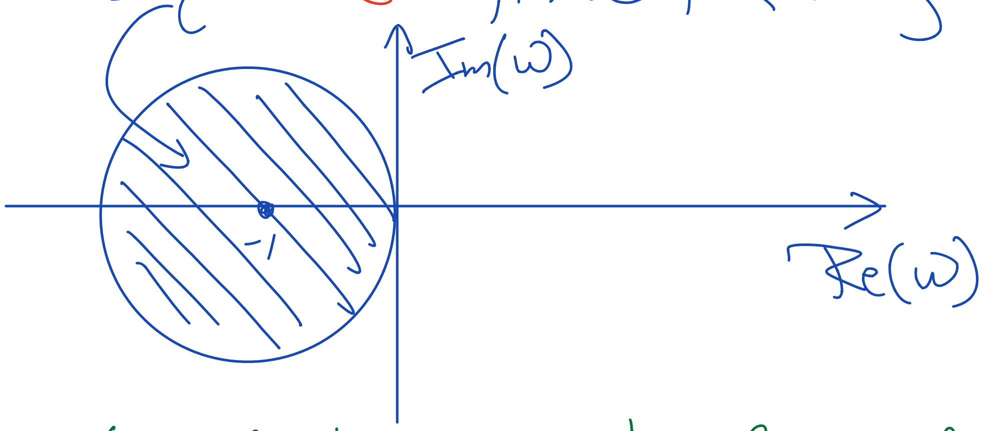
Euler's method

then $\phi(z) = (z-1) - \underbrace{h\lambda}_{}$

$= (z-1) - \omega \Rightarrow$ Root: $1+\omega$

$\Rightarrow$ Region of absolute stability

is $\{ \omega \in \mathbb{C} : |1+\omega| < 1 \}$



$$\left(|1+h\lambda| < 1 \iff |1+h\mu+ih\nu| < 1 \right.$$

$$\iff \sqrt{(1+h\mu)^2 + (h\nu)^2} < 1$$

$$1 + h^2\mu^2 + 2h\mu + h^2\nu^2 < 1$$

$$h^2(\mu^2+\nu^2) < -2h\mu$$

$$h < -2\mu / (\mu^2+\nu^2)$$

Practical implications

To use higher order methods, we'd like $w - \lambda h \in R$:= Region of abs. stab

When $x' = \lambda x$ is our ODE we just pick h so that $\lambda h \in R$

If you have a non-linear IVP

$$x' = f(t,x) \xrightarrow[\text{linear approx}]{} f(t,x) \approx \lambda_t x$$